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Feedback spreading controls for semilinear parabolic systems

Khalid Kassara

Department of Mathematics, University of Casablanca 1, P.O. Box 5366, Casablanca, Morocco

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Abstract

Distributed biophysical processes usually involve spreads in space, e.g., pollution, desertification in vegetation dynamics, etc. In this paper, based on the fact that spreadability of a distributed system reduces to a monotonicity problem, we show how feedback spreading control laws can be determined in semilinear parabolic systems. In case of affine dependence upon the control the use of a technique which combines saddle points and contraction mapping theorem yields a minimum "energy" feedback spreading control law. A mathematical example is examined in order to illustrate the derived results.

Keywords: Semilinear parabolic equation; Monotonicity; Optimal control; Constrained optimization; Fixed points

1. Introduction and problem setting

The aim of this paper is to investigate the problem of spreading control which is stated in [8,9] under several environmental motivations. Roughly speaking, it consists of determining controls which can achieve a spreadable system or, in other words, in such a manner that the excited system involves a family of spreading zones.

To be more precise let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with sufficiently smooth boundary $\partial\Omega$ and set $Q = \Omega \times (0, \infty[$. Let A be a second-order elliptic operator on Ω given in the form

$$A \doteq -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} + a_0(x) \tag{1.1}$$

E-mail address: kassara@facsc-achok.ac.ma (K. Kassara)

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with the smooth functions a_{ij}, a_i and a_0. Then, for convenient boundary data, it can be assumed that the operator $-A$ stands for an unbounded densely defined linear operator which generates a $\mathscr{C}_0$ analytic semigroup on $Z = L^2(\Omega)$, see [2,11].

We then consider the following semilinear parabolic control system:

$$\begin{aligned} &\frac{\partial z}{\partial t} + Az = \varphi(z, v) \quad \text{in } Q, \\ &z(x, 0) = z_0(x) \quad \text{in } \Omega, \end{aligned} \tag{1.2}$$

where $z_0 \in \mathscr{D}(A)$ and φ denotes a nonlinear operator which maps $Z \times V$ into Z with V another Hilbert space.

Further, let us consider the set-valued map:

$$\omega : \mathscr{D}(\omega) \subset Z \to 2^{\Omega}. \tag{1.3}$$

The problem we have to deal with is then: Find control $\bar{v}$ on a time interval $[0, t_1[$ such that

(i) system (1.2) with control $\bar{v}$ has a solution $\bar{z}$.

(ii) The zones $(\omega(\bar{z}(t))_{0 \leqslant t < t_1}$ are nondecreasing, i.e., $\omega(\bar{z}(t)) \subset \omega(\bar{z}(s))$ if $0 \leqslant t \leqslant s < t_1$.

The controls $\bar{v}$ are thereby called ω-spreading controls.

For instance, in tidal dynamics (see [4]), the pollution process may be modeled by the spreadability of the system which describes the pollution *concentration* where the map ω of Eq. (1.3) is taken as follows:

$$\omega(z) = \{x \in \Omega \mid z(x) > z_{\max}\} \tag{1.4}$$

in which $z_{\max}$ denotes a tolerance coefficient.

For practical reasons it would be of interest to consider feedback spreading controls in the form

$$v = \psi(z, \omega(z)). \tag{1.5}$$

The aim of this paper is to examine questions of existence and optimality of control laws (1.5). The main technique for the analysis involves the notion of monotonicity with respect to a preorder, see [1] and the references therein. This is because of condition (ii) which characterizes a spreading control.

In this paper we use the following definitions and notations: The set-valued map $Q : \mathscr{K} \mapsto Y$ where Y is a metric space is said to be *lower semicontinuous* if for each $z^0 \in \mathscr{K}$ and any sequence of elements z_n converging to z^0 then for each $y^0 \in Q(z^0)$, there exists a sequence of elements $y_n \in Q(z_n)$ which converges to y^0. The graph of Q is denoted by Graph(Q) and stands for the set $\{(z, y) \in \mathscr{K} \mid y \in Q(z)\}$. A selection of the map Q is a mapping $\xi : \mathscr{K} \mapsto Y$ which satisfies $\xi(z) \in Q(z)$ for every $z \in \mathscr{K}$. When Y stands for a Hilbert space then a mapping from Z to Y is said strongly–weakly continuous and denoted *st–we continuous*, if it maps strongly convergent sequences in Z into weakly convergent sequences in Y.

We organize the paper as follows: In Section 2 we give the basic results we have to use in this paper. They mainly concern monotonicity in infinite-dimensional spaces. Then Section 3 gives conditions of existence of feedback spreading controls and Section 4 is devoted to their optimality with respect to “energy”. In Section 5 we develop an illustrative example.

2. On monotonicity

Let Y be a separable Hilbert space and $\mathscr{K}$ a subset of Y. A set-valued map $P:\mathscr{K}\mapsto\mathscr{K}$ is said to be a preorder on $\mathscr{K}$ if it satisfies for each $z\in\mathscr{K}$,

$$z\in P(z) \quad \text{and} \quad y\in P(z)\Rightarrow P(y)\subset P(z).$$

Given $\bar{y}:[0,t_1(\mapsto\mathscr{K})]$ then it is said to be monotone with respect to the preorder P if

$$\bar{y}(s)\in P(\bar{y}(t)) \quad (0\leqslant t<s<t_1). \tag{2.1}$$

Numerous authors have been concerned with the problem of existence of monotone solutions with respect to a preorder in the general setting of differential inclusions, see for instance [1,5,6,12]. In the sequel, we give an adaptation of a result which is established in [6] to our context. Let $-B$ be an unbounded linear operator which generates a $\mathscr{C}_0$ semigroup $(T(t))_{t>0}$ on Y. Consider the semilinear abstract differential equation

$$\begin{aligned} &\dot{y}+By=g(y), \quad t\geqslant 0,\\ &y(0)=y_0, \end{aligned} \tag{2.2}$$

where $g:\mathscr{D}(B)\mapsto Y$ stands for a nonlinear operator. Also we need to consider the following tangential condition for each couple $(y,y_1)\in\mathscr{K}\times Y$:

$$\forall\delta>0, \quad \exists 0<h<\delta \quad \text{and} \quad \|p\|\leqslant\delta \text{ such that } T(h)y+h(y_1+p)\in P(y). \tag{2.3}$$

We therefore can define the tangential set

$$\mathscr{A}^P(y)\doteq\{y_1\,|\,(y,y_1) \text{ satisfies (2.3)}\}. \tag{2.4}$$

Hence we are ready to set the following result.

Theorem 2.1. *Let P be a preorder on $\mathscr{K}$ and assume the statements below to be satisfied*:

(i) *The semigroup $(T(t))_{t>0}$ is* compact.
(ii) *The operator $g|_{\mathscr{K}}$ is st–we continuous.*
(iii) *Graph(P) is closed in Y^2.*

Then the system (2.2) *has monotone solutions with respect to the preorder P for all initial data $y_0\in\mathscr{K}$ if and only if*

$$g(y)\in\mathscr{A}^P(y) \quad (y\in\mathscr{K}). \tag{2.5}$$

Remark 2.2. It should be useful to emphasize that since the semigroup $T(\cdot)$ is of class $\mathscr{C}_0$ it easily can be seen that if

$$\mathscr{K}\subset\mathscr{D}(B)$$

then the tangential condition (2.3) may be replaced by the following one:

$$\forall\delta>0, \quad \exists 0<h<\delta \quad \text{and} \quad \|p\|\leqslant\delta \quad \text{such that } y+h(y_1-By+p)\in P(y). \tag{2.6}$$

This has particular interest whenever the semigroup $T(\cdot)$ is not given explicitly.

Now we need to show the result below that we shall use in Section 4.

Proposition 2.3. *Assume the preorder P to have closed convex values; then the map*

$$z \in \mathscr{K} \to \mathscr{A}^P(z) \subset Y$$

is lower semicontinuous.

Proof. It is sufficient to show that for every $z \in Y$, the function $y \in \mathscr{K} \to d(z, \mathscr{A}^P(y))$ is upper semicontinuous in the sense of real functions, where d is the usual metric on Y. On the other hand, we know that if D is a closed convex set then the map

$$y \in D \to T_D(y) \subset Z$$

is lower semicontinuous, where $T_D(\cdot)$ stand for the cotangent set of D, see [7, Proposition 4.1]. Since P is a preorder, it follows that $P(u) \subset P(y)$ for every $u \in P(y)$ and it is easy to verify that

$$T_{P(u)}(y) \subset T_{P(y)}(y)$$

and from here

$$d(z, T_{P(y)}(y)) \leqslant d(z, T_{P(u)}(y)).$$

So, if P has convex closed values the mapping

$$y \mapsto d(z, T_{P(u)}(y))$$

is upper semicontinuous and from inequality above, we conclude that the mapping $y \in Y \to d(z, T_{P(y)}(y))$ is also upper semicontinuous. □

3. Characterization of feedback spreading controls

Let us turn to the control system (1.2) with state space

$$Z = L^2(\Omega).$$

Assume the linear operator A to be as in (1.1) and denote by $S(\cdot)$ the semigroup that it generates on Z. Let ω be the map to be spread and $\mathscr{S}$ be a subset in $\mathscr{D}(\omega)$. Then a feedback spreading control law may be defined as follows.

Definition 3.1. The mapping $\varsigma : \mathscr{S} \mapsto V$ is said to be a feedback spreading control law for system (1.2) if for all initial data $z_0 \in \mathscr{S}$ the control $\bar{v} = \varsigma(\bar{z})$ is a spreading control.

Now let us define

$$P_\omega(z) \doteq \{y \in \mathscr{S} \mid \omega(y) \supset \omega(z)\} \tag{3.1}$$

which obviously stands for a preorder on $\mathscr{S}$. It follows that condition (ii) in Section 1 which characterizes a spreading control may be rewritten as follows:

$$\bar{z}(s) \in P_\omega(\bar{z}(t)).$$

Consequently, ω-spreading controls for system (1.2) are such that the controlled system has monotone solutions with respect to the preorder P_ω. Hence, we can proceed to use Theorem 2.1 in order to derive our main existence result. To this end we need first to set

$$\mathscr{M}^\omega(\cdot) \doteq \mathscr{A}^{P_\omega}(\cdot), \tag{3.2}$$

where the map $\mathscr{A}^{(\cdot)}$ is as in (2.4). Therefore, for each $z \in \mathscr{S}$ the elements $z_1 \in \mathscr{M}^\omega(z)$ are given by the tangential condition

$$\begin{aligned} &\forall \delta > 0, \quad \exists 0 < h < \delta \quad \text{and} \quad \|p\| \leqslant \delta \text{ such that } S(h)z + h(z_1 + p) \in \mathscr{S}, \\ &\omega(S(h)z + h(z_1 + p)) \supset \omega(z). \end{aligned} \tag{3.3}$$

Now let us introduce the *feedback map* as follows:

$$\mathscr{F}^\omega(z) \doteq \{v \in V \mid \varphi(z, v) \in \mathscr{M}^\omega(z)\}. \tag{3.4}$$

Also we need let

$$\Sigma_\omega \doteq \{(y, z) \in \mathscr{S}^2 \mid \omega(y) \supset \omega(z)\}. \tag{3.5}$$

Hence we are ready to prove the following result.

Proposition 3.2. *Assume the set Σ_ω to be closed and the semigroup $S(\cdot)$ to be compact. Let $\varsigma : \mathscr{S} \mapsto V$ be such that the mapping $\varphi(\cdot, \varsigma(\cdot))$ is st–we continuous on $\mathscr{S}$. Then ς is a feedback spreading control law if and only if it is a* selection *of the feedback map $\mathscr{F}^\omega$.*

Proof. We have to use Theorem 2.1. Take $Y = Z$, $\mathscr{K} = \mathscr{S}$, $B = A$, $g = \varphi(\cdot, \varsigma(\cdot))$ and $P = P_\omega$ as given by (3.1). Since the operator $-A$ generates a compact semigroup then condition (i) is satisfied. We easily can see that all the assumptions in the above cited theorem hold. □

Note that in the above proposition, there are no continuity assumptions on φ nor on ς. Only it is required that the mapping $\varphi(\cdot, \varsigma(\cdot)$ is st–we continuous on $\mathscr{S}$.

Also it should be useful to stress that the expression ψ of the feedback law as presented in Eq. (1.5) actually holds. In fact, for each subset σ in Ω we can take $\psi(z, \sigma)$ as a selection of the map $\mathscr{F}^\omega_\sigma$ defined as the map $\mathscr{F}^\omega$ in which $\omega(z)$ is replaced by σ in the tangential condition (3.3).

4. An optimal spreading control problem

Until recently the problem of spreading control with minimum "energy" has been investigated only for the linear case (see [9]). Therein, although the approach leads to an optimality system involving a Riccati differential equation, it only provides approximate solutions which unfortunately are not easy to compute because of the infinite-dimensional setting.

In this section we deal with the minimum eneregy problem in a more general context. Since feedback spreading controls are given by a selection procedure of the feedback map $\mathscr{F}^\omega(\cdot)$ then finding minimum energy feedback spreading laws reduces to the parametrized constrained optimization problem below:

$$\text{For each } z \in \mathscr{S} \text{ find } \min_{v \in \mathscr{F}^\omega(z)} \|v\|^2. \tag{4.1}$$

Thus, the mapping which sends each $z \in \mathscr{S}$ to the solution of problem (4.1) can define the "minimum energy" spreading control law providing that conditions of Proposition 3.2 hold. By virtue of (3.4) problem (4.1) is equivalent to

$$\text{For each } z \in \mathscr{S} \text{ find } \min_{\varphi(z,v)\in\mathscr{M}^{\omega}(z)} \|v\|^2 \tag{4.2}$$

Next we need to impose the following hypotheses:

H_1 $-A$ generates a compact semigroup $S(\cdot)$ on Z.
H_2 System (1.2) is affine in the controls, i.e.,

$$\varphi(z,v) = f(z) + G(z)v \quad (z \in \mathscr{S}, v \in V), \tag{4.3}$$

where f and G act in $\mathscr{S}$ and respectively have images in Z and $\mathscr{L}(V,Z)$.
H_3 The map $\omega : \mathscr{S} \mapsto 2^{\Omega}$ satisfies
(i) $\Sigma_\omega \doteq \{(y,z) \in \mathscr{S}^2 \mid \omega(y) \supset \omega(z)\}$ is closed in $\mathscr{S}^2$,
(ii) $\omega(\alpha y + \beta \bar{y}) \supset \omega(y) \cap \omega(\bar{y})$ $\forall y, \bar{y} \in \mathscr{S}$ and $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = 1$.

We begin by proving the following basic result which concerns the map $\mathscr{M}^{\omega}(\cdot)$ given by Eq. (3.4).

Lemma 4.1. *We have*
(a) *For each $z \in \mathscr{S}$ the set $\mathscr{M}^{\omega}(z)$ is closed and convex.*
(b) *The set-valued map $\mathscr{M}^{\omega}$: $z \in \mathscr{S} \mapsto \mathscr{M}^{\omega}(z) \subset Y$ is lower semicontinuous.*

Proof. To show (a): let P_ω as given in Eq. (3.1) then since Σ_ω is closed then so is the subset $P_\omega(z)$ for each $z \in \mathscr{S}$. Consequently, from expressions (3.2) and (3.3) we get

$$\mathscr{M}^{\omega}(z) = \mathrm{cl}\left(\bigcup_{h>0} \frac{1}{h}[P_\omega(z) - S(h)z] \right) \quad (z \in \mathscr{S}).$$

Therefore $\mathscr{M}^{\omega}(z)$ is closed for each $z \in \mathscr{S}$. To show that $\mathscr{M}^{\omega}(z)$ is convex it suffices to observe that the condition H_3(ii) implies that $P_\omega(z)$ is convex for each $z \in \mathscr{S}$ and so is $\mathscr{M}^{\omega}(z)$ as the closure of a convex subset.

For the lower semicontinuity of the map $\mathscr{M}^{\omega}$, this easily may be seen by using Proposition 2.3 with $P = P_\omega$. $\square$

The result we show below presents the optimization technique we will use for solving the problem (4.2) under hypothesis H_2. It mainly consists of a combination of a saddle point method and the contraction mapping theorem; see [10] for more details on the method. For that we need to denote by $\langle\, ; \rangle$ the scalar product on a Hilbert space which is clear from the context. We also denote by $\pi_{\mathscr{K}}(\cdot)$ the projector of best approximation on a closed convex subset $\mathscr{K}$ and by B^* the adjoint of a bounded linear operator B between two Hilbert spaces.

Lemma 4.2. *Let $f \in Z$ and $\mathscr{M}$ be a closed convex subset of Z. Consider a linear operator $G \in \mathscr{L}(V,Z)$ satisfying the following condition.*

$$\|G^* y\|^2 \geqslant m\|y\|^2 \quad (y \in Z), \tag{4.4}$$

where $m > 0$. Then the minimization problem

$$\min_{Gv+f\,\in\,\mathcal{M}} \|v\|^2 \tag{4.5}$$

has a unique solution v_0 given by

$$v_0 = G^*R^{-1}(y_0 - f), \tag{4.6}$$

where $R \doteq GG^*$ and y_0 is given by the fixed point equation

$$y_0 = \pi_{\mathcal{M}}[(1 - \rho R^{-1})y_0 + \rho R^{-1} f] \tag{4.7}$$

for some $\rho > 0$.

Proof. Since $\mathscr{C} = \{v \in V \mid Gv + f \in \mathcal{M}\}$ is a nonempty closed convex subset in V then the problem (4.5) has a unique solution which is $v_0 = \pi_{\mathscr{C}}(0)$. To compute v_0 a proper method may be provided by the Lagrangian functional below:

$$L(v, y, \mu) = \tfrac{1}{2}\|v\|^2 + \langle Gv + f - y; \mu\rangle \quad (v \in V, y \in \mathscr{T}, \mu \in Z).$$

In fact, it can be easily shown that if (u_0, y_0, μ_0) is a saddle point for L, i.e.,

$$\max_{\mu\,\in\,Z} L(u_0, y_0, \mu) = L(u_0, y_0, \mu_0) = \min_{v\,\in\,V, y\,\in\,\mathcal{M}} L(v, y, \mu_0),$$

then u_0 is a solution to the problem (4.2) and by unicity $u_0 = v_0$. Now since both L and $\mathcal{M}$ are convex then the saddle point (v_0, y_0, μ_0) is characterized by

$$\frac{\partial L}{\partial v}(v_0, y_0, \mu_0) = 0$$

$$\left\langle \frac{\partial L}{\partial y}(v_0, y_0, \mu_0); y - y_0 \right\rangle \geqslant 0 \quad (y \in \mathcal{M}),$$

$$\frac{\partial L}{\partial \mu}(v_0, y_0, \mu_0) = 0$$

so that we get

$$v_0 + G^*\mu_0 = 0$$

$$\langle \mu_0; y - y_0\rangle \leqslant 0 \quad (y \in \mathcal{M}),$$

$$Gv_0 + f = y_0$$

and therefore in an equivalent way, we have $v_0 = -G^*\mu_0$ where (μ_0, y_0) solves the following system:

$$\begin{aligned} &-R\mu_0 + f = y_0 \\ &\langle \rho\mu_0, y - y_0\rangle \leqslant 0 \quad (y \in \mathcal{M},\ \rho > 0). \end{aligned} \tag{4.8}$$

Now by (4.4), the operator R is invertible since it is symmetric and coercive. Using $\pi_{\mathcal{M}}$ yields

$$v_0 = G^*R^{-1}(y_0 - f),$$

$$y_0 = \pi_{\mathcal{M}}[(1 - \rho R^{-1})y_0 + \rho R^{-1} f] \quad (\rho > 0).$$

Now in order to complete the proof it remains to show that the mapping

$$\Phi_\rho : \mathscr{M} \mapsto \mathscr{M},$$
$$y \mapsto \pi_{\mathscr{M}}[(1-\rho R^{-1})y + \rho R^{-1} f]$$

has a fixed point for some $\rho > 0$. Indeed we get

$$\|\Phi_\rho(y) - \Phi_\rho(\bar{y})\|^2 \leqslant \|(1-\rho R^{-1})e\|^2$$
$$= \|e\|^2 - 2\rho\langle R^{-1}e; e\rangle + \rho^2\|R^{-1}e\|^2,$$

where $y, \bar{y} \in \mathscr{M}$ and $e = y - \bar{y}$.

Now since the operator R^{-1} is coercive we have for some $m' > 0$,

$$\langle R^{-1}y; y\rangle \geqslant m'\|y\|^2 \quad (y \in Z)$$

then it follows that

$$\|\Phi_\rho(y) - \Phi_\rho(\bar{y})\|^2 \leqslant (1 - 2\rho m' + \rho^2\|R^{-1}\|^2)\|y - \bar{y}\|^2.$$

Therefore Φ_ρ is a contraction for $\rho < 2m'/\|R^{-1}\|^2$ and thereby it has a unique fixed point y_0 which belongs to $\mathscr{M}$. □

Now let us prove the main result in this section.

Theorem 4.3. *Assume that the following conditions hold*:

(i) *For each $z \in \mathscr{S}$ the operator $G(z)$ satisfies the following coercivity condition*,

$$\|G^*(z)y\|^2 \geqslant m_z\|y\|^2 \quad (y \in Z) \tag{4.9}$$

where the coefficient $m_z > 0$ is such that: *For each $\alpha > 0$ there exists $M > 0$ such that*

$$z \in \mathscr{S},\ \|z\| < \alpha \Rightarrow m_z > M. \tag{4.10}$$

(ii) *The mapping $f : \mathscr{S} \mapsto Z$ is st–we continuous.*

(iii) *The control operator $G(\cdot)$ verifies*

$$\left.\begin{array}{l} z_n \to z \text{ (strong in } Z) \\ v_n \to v \text{ (weak in } V) \end{array}\right\} \Rightarrow G(z_n)v_n \to G(z)v \text{ (weak in } Z).$$

Then there exists a unique feedback spreading control law ς_* which solves problem (4.2). *It is given for each $z \in \mathscr{S}$ by*

$$\varsigma_*(z) = G^*(z)R^{-1}(z)(\phi_s(z) - f(z)), \tag{4.11}$$

where $R(\cdot) = G(\cdot)G^*(\cdot)$ and $\phi_s = \phi_s(z)$ satisfies the fixed point equation

$$\phi_s = \pi_{\mathscr{M}^\omega(z)}[(1 - \rho R^{-1}(z))\phi_s + \rho R^{-1}(z)f(z)] \tag{4.12}$$

for some $\rho > 0$. Furthermore the mapping $\varsigma_* : \mathscr{S} \mapsto V$ is st–we continuous.

Proof. We have to solve for each $z \in \mathscr{S}$,

$$\min \|v\|^2 \text{ with constraints: } G(z)v + f(z) \in \mathscr{M}^{\omega}(z) \tag{4.13}$$

Indeed let $z \in \mathscr{S}$ and take in Lemma 4.2 $G = G(z); f = f(z)$ and $\mathscr{M} = \mathscr{M}^{\omega}(z)$. Condition (4.9) obviously implies the coercivity condition which is required in that lemma. Moreover, by Lemma 4.1(a) the set $\mathscr{M}^{\omega}(z)$ is closed and convex. It follows that the problem (4.13) has a unique solution $v_* = \varsigma_*(z)$ for each $z \in \mathscr{S}$ and thus expressions (4.11) and (4.12) respectively follow from (4.6) and(4.7).

Now in order to complete the proof it remains to show, in accord with Proposition 3.2, that the law ς_* is such that the mapping $f + G(\cdot)\varsigma_* = \phi_s$ is st–we continuous.

Indeed let $(z_n)_n$ be a sequence with (strong) limit $z \in \mathscr{S}$; then for each z_n, applying the optimality system (4.8) yields

$$\begin{aligned} &R(z_n)\mu_0(z_n) = f(z_n) - \phi_s(z_n) \\ &\langle \mu_0(z_n), y - \phi_s(z_n)\rangle \leqslant 0 \quad (y \in \mathscr{M}^{\omega}(z_n)). \end{aligned} \tag{4.14}$$

Now let $y \in \mathscr{M}^{\omega}(z)$. We know that the map $\mathscr{M}^{\omega}(\cdot)$ is lower semicontinuous on $\mathscr{S}$ (see Lemma 4.1b); therefore there exists a sequence $(y_n)_n$ which converges to y and satisfies

$$y_n \in \mathscr{M}^{\omega}(z_n) \quad (\text{for each } n).$$

Hence by letting $y \doteq y_n$ in Eq. (4.14) we get

$$\langle \mu_0(z_n); R(z_n)\mu_0(z_n)\rangle \leqslant \langle \mu_0(z_n); f(z_n) - y_n\rangle \quad (\text{for each } n)$$

and then by using (4.9) it follows that

$$m_{z_n}\|\mu_0(z_n)\|^2 \leqslant \|G^*(z_n)\mu_0(z_n)\|^2 \leqslant \langle \mu_0(z_n); f(z_n) - y_n\rangle \quad (\text{for each } n).$$

Consequently, by condition (4.10), it follows that the sequence $(\mu_0(z_n))_n$ is bounded. It therefore has a subsequence $(\mu_0(z_q))_q$ which is weakly convergent to $\bar{\mu}_0 \in Z$. Hence (ii) and (iii) imply that

$$-R(z_q)\mu_0(z_q) + f(z_q) \to -R(z)\bar{\mu}_0 + f(z) = \bar{\phi}_s \quad (\text{weak in } Z)$$

Therefore

$$\phi_s(z_q) \to \bar{\phi}_s \quad (\text{weak in } Z).$$

Even, considering (4.14) with y_q instead of y and passing to the limit yields

$$\langle \bar{\mu}_0, y - \bar{\phi}_s\rangle \leqslant 0 \quad (y \in \mathscr{M}^{\omega}(z)).$$

It follows that $(\bar{\mu}_0, \bar{\phi}_s)$ is a solution of the optimality system and by unicity we get $\bar{\mu}_0 = \mu_0(z)$ and $\bar{\phi}_s = \phi_s(z)$. Therefore the sequences $(\phi_s(z_n))_n$ and $(\mu_0(z_n))_n$ respectively converge weakly to $\phi_s(z)$ and $\mu_0(z)$ and so does the sequence $(\varsigma_*(z_n))_n$ whose limit is $\varsigma_*(z)$. This shows that the mappings ϕ_s and ς_* are st–we continuous on $\mathscr{S}$. $\square$

One advantage of the technique employed in the above theorem resides in the resulting computational issues. In fact, the sequence $(\phi_s^q(\cdot))_q$ may be updated by the fixed point iterative scheme below:

Step 1: Choose $\phi_s^0(\cdot)$ such that

$$\phi_s^0(z) \in \mathscr{M}^\omega(z) \quad (z \in \mathscr{S})$$

Step 2: Iterate the sequence $\phi_s^q(\cdot)$ as follows:

- At the level q let

$$y_q(z) = R^{-1}(z)(\phi_s^q(z) - f(z)) \quad (z \in \mathscr{S}).$$

- Then compute

$$\phi_s^{q+1}(z) = \pi_{\mathscr{M}^\omega(z)}(\phi_s^q(z) - \rho y_q(z)) \quad (\text{for small } \rho > 0).$$

Step 3: Let q_f (sufficiently large) be the final level in Step 2, then the corresponding suboptimal feedback spreading control law is given by

$$\varsigma_f(z) = G^*(z) y_{q_\mathrm{f}}(z).$$

Concerning the above algorithm we can note the following remarks:

- The final level in Step 3 can be taken such that there hold the following criterion:

$$\|\phi_s^{q+1}(z) - \phi_s^q(z)\| \leqslant \varepsilon \quad (z \in \mathscr{S})$$

  for a small given $\varepsilon > 0$.
- For instance, we can take in Step 1, $\phi_s^0(z) = Az$ for each $z \in \mathscr{S}$.
- The coefficient ρ in Step 2 may be updated with respect to z. For each $z \in \mathscr{S}$ it suffices to take ρ such that

$$0 < \rho < \frac{2m_z'}{\|R^{-1}(z)\|^2}.$$

5. An example

Let Ω be an open bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Consider the semilinear parabolic control system:

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z = \sum_{i=1}^{3} z \frac{\partial z}{\partial x_i} + a(x,z)u(x,t) + b(x)w(t), \\ &z(t)|_{\partial\Omega} = 0, \quad (x \in \Omega;\ t > 0), \end{aligned} \tag{5.1}$$

where Δ is the Laplacian operator given by

$$\Delta z = \sum_{i=1}^{3} \frac{\partial^2 z}{\partial x_i^2} \quad (x = (x_1, x_2, x_3) \in \Omega).$$

The system is controlled by $v(t) = (u(\cdot,t), w(t))$. The corresponding control space is then $V = L^2(\Omega) \times \mathbb{R}$. It is well known that the operator $A = -\Delta$ with domain $\mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ generates a compact analytic semigroup on $Z = L^2(\Omega)$, see [2]. Furthermore, the system is affine in the controls as required in Hypothesis H_2 where

$$G(z)v = a(\cdot, z)u(\cdot) + b(\cdot)w \tag{5.2}$$

and

$$f(z) = \sum_{i=1}^{3} z\frac{\partial z}{\partial x_i} \tag{5.3}$$

for each $z \in Z$ and $v = (u(\cdot), w) \in V$. Now let $\mathscr{S}$ be a closed subset in Z such that

$$\mathscr{S} \subset \mathscr{D}(A) \cap L^\infty(\Omega).$$

Then take the map ω as follows:

$$\omega(z) = \{x \in \Omega \mid \|x\|_n > \|z\|\} \quad (z \in \mathscr{S}). \tag{5.4}$$

Then we can show the following result.

Lemma 5.1. *We have the statements below*:

(i) *The hypothesis* H_3 *holds true.*

(ii) *The mapping* f *of Eq.* (5.3) *is st–we continuous on* $\mathscr{S}$.

(iii) *The set* $\mathscr{M}^\omega(\cdot)$ *defined by* (3.2) *is given by*

$$\mathscr{M}^\omega(z) = \{y \in Z \mid \langle y - Az; z\rangle \leqslant 0\} \quad (z \in \mathscr{S}). \tag{5.5}$$

Proof. From (5.4) we have for each $(y, z) \in \mathscr{S}^2$,

$$\omega(y) \supset \omega(z) \Leftrightarrow \|z\| \geqslant \|y\|. \tag{5.6}$$

Then it follows that the set $\Sigma_\omega = \{(y, z) \in \mathscr{S}^2 \mid \omega(y) \supset \omega(z)\}$ is closed. The convexity condition in hypothesis H_3(ii) also is obvious.

To show (ii) let $(z_n)_n$ be a sequence in $\mathscr{S}$ with (strong) limit $z \in \mathscr{S}$. Let $\phi \in \mathscr{C}_0^\infty(\Omega)$; then the fact that $z_n \in H^1(\Omega) \cap L^\infty(\Omega)$ implies that

$$\int_\Omega z_n \frac{\partial z_n}{\partial x_i}\phi = -\frac{1}{2}\int_\Omega z_n^2 \frac{\partial \phi}{\partial x_i} \quad (n \geqslant 1;\ i = 1, 2 \text{ or } 3).$$

Then the Dominated Convergence Theorem applied to subsequences of $(z_n)_n$ which converge almost everywhere to z yields

$$-\frac{1}{2}\int_\Omega z_n^2 \frac{\partial \phi}{\partial x_i} \to -\frac{1}{2}\int_\Omega z^2 \frac{\partial \phi}{\partial x_i} \quad (i = 1, 2 \text{ or } 3),$$

whence we obtain

$$\langle f(z_n), \phi\rangle \to \langle f(z), \phi\rangle \quad (\phi \in \mathscr{C}_0^\infty(\Omega))$$

and then by density the proof of (ii) is complete.

Concerning the statement (iii), by Remark 2.2 we get for each $z \in \mathscr{S}$,

$$y \in \mathscr{M}^\omega(z) \Leftrightarrow \{\forall \delta > 0, \exists h < \delta, \|p\| < \delta \text{ such that } \omega(z + h(y - Az + p)) \supset \omega(z).$$

Then taking in account (5.6) yields

$$y \in \mathscr{M}^{\omega}(z) \Leftrightarrow \{\forall \delta > 0, \exists h < \delta, \|p\| < \delta) \text{ such that } \|z + h(y - Az + p)\|^2 \leqslant \|z\|^2$$

and finally we obtain

$$y \in \mathscr{M}^{\omega}(z) \Leftrightarrow \langle y - Az; z\rangle \leqslant 0. \qquad \square$$

Now let us suppose the following assumptions:

A_1 For each $z \in \mathscr{S}$ there exist k_z and K_z as in Eq. (4.10) such that $k_z \leqslant |a(\cdot, z)| \leqslant K_z$ on Ω.
A_2 The mapping $z \in \mathscr{S} \mapsto a(\cdot, z) \in L^{\infty}(\Omega)$ is continuous.
A_3 $b \in L^{\infty}(\Omega)$.

Then we can show the result below.

Lemma 5.2. *Under assumptions A_1–A_3, conditions* (i) *and* (iii) *in Theorem* 4.3 *are satisfied.*

Proof. Indeed to show that condition (i) of Theorem 4.3 is satisfied it suffices to see that

$$G^*(z)y = (a(\cdot, z)y, \langle y; b\rangle) \quad (y \in Z) \tag{5.7}$$

and therefore the inequality (4.9) holds true with $m_z = k_z$ and

$$M_z = K_z + \|b\|_{L^{\infty}(\Omega)}.$$

Now from assumption A_2 we easily can see that condition (iii) of Theorem 4.3 also is satisfied. It suffices to see that for each sequences z_n and $v_n = (u_n(\cdot), w_n)$ we have by using (5.2):

$$\langle G(z_n)v_n - G(z); y\rangle = \langle (a(\cdot, z_n) - a(\cdot, z))u_n; y\rangle + \langle u_n - u; a(\cdot, z)y\rangle + (w_n - w)\langle b; y\rangle$$

for each $y \in Z$. $\square$

Consequently, from Lemmas 5.1 and 5.2 it follows that each of the conditions of Theorem 4.3 are satisfied. Therefore there exists a unique minimum energy feedback spreading control law

$$\varsigma = (\varsigma_1, \varsigma_2) : \mathscr{S} \mapsto V$$

which is given by formulae (4.11) and (4.12). So let us proceed to compute the operators which appear in these formulae. First, from (5.2) and (5.7) we get

$$R(z)y = G(z)G^*(z)y = a^2(\cdot, z)y + \langle y, b\rangle b \quad (y \in Z,\ z \in \mathscr{S}).$$

Then a direct computation of $R^{-1}(\cdot)$ yields

$$R^{-1}(z)y = \frac{y}{a^2(\cdot, z)} - \frac{\langle y, c\rangle b}{1 + \langle b, c\rangle} \quad (y \in Z,\ z \in \mathscr{S}) \tag{5.8}$$

with

$$c = c(\cdot, z) = \frac{b}{a^2(\cdot, z)}.$$

Now, by considering formula (5.5), the operator of best approximation $\pi_{\mathscr{M}^\omega(\cdot)}$ may be given as follows:

$$\pi_{\mathscr{M}^\omega(z)}(y) = \begin{cases} y - \dfrac{\max[\langle y - Az; z\rangle, 0]}{\|z\|^2} z & \text{if } z \neq 0, \\ y & \text{if } z = 0. \end{cases} \tag{5.9}$$

Consequently, the sequences of the algorithm which ends Section 4 can be executed as follows: In Step 2, y_q can be computed by using formula (5.8) and then ϕ_s^{q+1} by considering Eq. (5.9).

In Step 3, expression (5.7) yields

$$\varsigma_1(z) = a(\cdot, z) y_{q_{\mathrm{f}}}(z) \quad (z \in \mathscr{S}),$$

$$\varsigma_2(z) = \langle b, y_{q_{\mathrm{f}}}(z)\rangle \quad (z \in \mathscr{S}).$$

6. Conclusion

In the present paper we have shown how the problem of spreading control can be solved for a given semilinear parabolic system. In the case where the system in affine in the controls, we have established an easily implemented algorithm which enabled us to approximate the minimum "energy" feedback spreading control law. This has been illustrated through a mathematical example which consists of a semilinear parabolic control equation.

Natural directions for further work include:

- The study of the maximum speed spreading control problem which has been stated in [9].
- The study of semilinear hyperbolic systems which do not involve compact semigroups. The reason is that numerous processes in which one can observe spreading phenomena are of hyperbolic kind, see [3,4].
- The numerical simulation of the derived feedback spreading laws.

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